数理解析研究所講究録
第 1906 巻 2014 年 92-94

A new necessary and sufficient condition for the Egoroff theorem in non-additive measure theory

Masayuki Takahashi, Toshiaki Murofushi, Shin Asahina
Department of Computational Intelligence and Systems Sciences,
Tokyo Institute of Technology
E-mail:masayuki@fz.dis.titech.ac.jp

Abstract: This paper is a brief summary of "M.Takahashi, T.Murofushi, S.Asahina, A new necessary and sufficient condition for the Egoroff theorem in non-additive measure theory, Fuzzy Sets and Systems, to appear." This paper states that a newly defined condition, called condition (M), is a necessary and sufficient condition for the Egoroff theorem in non-additive measure theory. The existing necessary and sufficient conditions for the Egoroff theorem are described by a doubly-indexed sequence of measurable sets, while condition (M) is described by a singly-indexed sequence of measurable sets.

1 INTRODUCTION

Since Sugeno [5] introduced the concept of non-additive measure, which he called a fuzzy measure, non-additive measure theory has been constructed along the lines of the classical measure theory [1, 4, 7]. Generally, theorems in the classical measure theory no longer hold in non-additive measure theory, so that to find necessary and/or sufficient conditions for such theorems to hold is very important for the construction of non-additive measure theory.

The Egoroff theorem, which asserts that almost everywhere convergence implies almost uniform convergence, is one of the most important convergence theorems in the classical measure theory. In non-additive measure theory, this theorem does not hold without additional conditions. Necessary and sufficient conditions for the Egoroff theorem to hold in non-additive measure theory have been proposed, for instance, the Egoroff condition [3] or a certain condition called condition (E) in [2]. However both conditions are complex since they are described by a doubly-indexed sequence of measurable sets. This paper gives a new simpler condition (M) which is described by a singly-indexed sequence of measurable sets and shows that condition (M) is equivalent to the Egoroff condition.

This paper is a brief summary of a forthcoming paper [6].

2 Preliminaries

Throughout the paper, $(X,\mathscr{S})$ is assumed to be a measurable space and $\mathbb{N}$ denotes the set of positive integers. In addition, every measurable function f is assumed to be finite real valued, i.e., $-\infty < f(x) < \infty$ for all $x \in X$.

Definition 1 A *non-additive measure* on $\mathscr{S}$ is a set function $\mu : \mathscr{S} \to [0,\infty]$ satisfying the following two conditions:

(i) $\mu(\emptyset) = 0$,

(ii) $A,B \in \mathscr{S},\ A \subset B \ \Rightarrow\ \mu(A) \le \mu(B)$.

Unless stated otherwise, all subsets are supposed to belong to $\mathscr{S}$ and μ is assumed to be a non-additive measure on $\mathscr{S}$.

Definition 2 Let $\{f_n\}$ be a sequence of measurable functions, and f be a measurable function.

(i) $\{f_n\}$ is said to converge to f *almost everywhere*, written $f_n \xrightarrow{\text{a.e.}} f$, if there exists N such that $\mu(N) = 0$ and $\{f_n(x)\}$ converges to $f(x)$ for all $x \in X \setminus N$.

(ii) $\{f_n\}$ is said to converge to f *almost uniformly*, written $f_n \xrightarrow{\text{a.u.}} f$, if for every $\varepsilon > 0$ there exists N_ε such that $\mu(N_\varepsilon) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on $X \setminus N_\varepsilon$.

Definition 3 μ is said to satisfy *the Egoroff condition* if, for every doubly-indexed sequence $E_{m,n}$ such that $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$ and $\mu(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}) = 0$, and for every positive number ε, there exists a sequence $\{n_m\}$ of positive integers such that $\mu(\bigcup_{m=1}^{\infty} E_{m,n_m}) < \varepsilon$. [3]

Murofushi et al. [3] show that the Egoroff condition is a necessary and sufficient condition for the Egoroff theorem, i.e., the Egoroff condition is satisfied iff almost everywhere convergence implies almost uniform convergence.

3 The Egoroff theorem

In this section, we define condition (M), which is equivalent to the Egoroff condition.

Definition 4 μ is said to satisfy *condition (M)* if $\mu(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i) = 0$ implies that for every positive number ε there exists a sequence $\{m_n\}$ of positive integers such that $\mu(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{m_n} E_i) < \varepsilon$.

The following lemma is necessary to show that condition (M) is equivalent to the Egoroff condition.

Lemma 1 *Let $\{E_{m,n}\}$ be a doubly-indexed sequence, $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, and $\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n} = \emptyset$. Then there exists a sequence $\{A_n\}$ such that $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \emptyset$, and for every strictly increasing sequence $\{k_n\}$ of positive integers, there exists a non-decreasing sequence $\{n_i\}$ of positive integers such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$.*

We can show Lemma 2, which is a generalization of Lemma 1.

Lemma 2 *For every doubly-indexed sequence $E_{m,n}$ such that $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, there exists a sequence $\{A_n\}$ of sets such that $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$, and for every strictly increasing sequence $\{k_n\}$, there exists a non-decreasing sequence $\{n_i\}$ such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$.*

It is not difficult to show that the Egoroff condition implies condition (M), while it is not easy to show the inverse implication. If it holds that for every doubly-indexed sequence $\{E_{m,n}\}$ such that $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, there exists a nondecreasing sequence $\{n_i\}$ of positive integers such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$, then that condition (M) implies the Egoroff condition is obvious. Assume that the above if-part is true, and condition (M) is satisfied. Let $\{E_{m,n}\}$ be a doubly-indexed sequence, $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, and $\mu(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}) = 0$. Since there exists a nondecreasing sequence $\{n_i\}$ of positive integers such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$, for each positive integer i, let $A_i = E_{i,n_i}$. Then, the Egoroff condition is satisfied obviously. However, it does not necessarily hold that for every doubly-indexed sequence $\{E_{m,n}\}$ such that $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, there exists a nondecreasing sequence $\{n_i\}$ of positive integers such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$.

By using Lemma 2, we can show the next theorem.

Theorem 1 *Condition (M) is equivalent to the Egoroff condition.*

4 Concluding remarks

In this paper, we have defined condition (M), which is a necessary and sufficient condition for the Egoroff theorem with respect to non-additive measure and is described by a singly-indexed sequence of measurable sets.

It seems that it is easier to treat condition (M) than the Egoroff condition since condition (M) is described by a singly-indexed sequence of measurable sets. However, concrete usefulness and potential applicability are not known. These are topics for future research.

REFERENCES

[1] D. Denneberg, *Non-additive Measure and Integral*, 2nd ed., Kluwer, Dordrecht, 1997.

[2] J. Li, A further investigation for Egoroff's theorem with respect to monotone set functions, *Kybernetika*, 39 (2003) 753–760.

[3] T. Murofushi, K. Uchino, and S. Asahina, Conditions for Egoroff's theorem in non-additive measure theory, *Fuzzy Sets and Systems*, 146 (2004) 135–146.

[4] E. Pap, *Null-additive Set Functions*, Kluwer, Dordrecht, 1995.

[5] M. Sugeno, *Theory of Fuzzy Integrals and Its Applications*, Doctoral Thesis, Tokyo Institute of Technology, 1974.

[6] M.Takahashi, T.Murofushi, and S.Asahina, A new necessary and sufficient condition for the Egoroff theorem in non-additive measure theory, *Fuzzy Sets and Systems*, to appear.

[7] Z. Wang and G.J. Klir, *Fuzzy Measure Theory*, Plenum, New York, 1992.